1. Let X be a topological space and let $\mathcal{B}$ be a base for the topology $\mathcal{T}$ of X. Let $Ens_{\mathcal{B}}$ denote the category of $\mathcal{B}$-sheaves of sets on $\mathcal{B}$ and let $Ens_{\mathcal{T}}$ denote the category of sheaves of sets on X. Prove that the restriction functor $Ens_{\mathcal{T}} \to Ens_{\mathcal{B}}$ is an equivalence of categories.

2. Let X be a topological space and let $\mathcal{U}$ be an open covering of X. By a *$\mathcal{U}$-sheaf* on X we mean a family of sheaves F_U for each $U \in \mathcal{U}$. By *descent data* on a $\mathcal{U}$-sheaf we mean a family of isomorphisms

$$\epsilon_{U,V} : F_{U_{|U\cap V}} \to F_{V_{|V\cap U}}$$

for each $(U, V) \in \mathcal{U} \times \mathcal{U}$, satisfying the following conditions (collectively called the "cocycle condition"):

   (a) $\epsilon_{U,V}{}_{|U\cap V\cap W} \circ \epsilon_{V,W}{}_{|U\cap V\cap W} = \epsilon_{U,W}{}_{|U\cap V\cap W}$ for $(U, V, W) \in \mathcal{U} \times \mathcal{U} \times \mathcal{U}$.
   (b) $\epsilon_{U,V} \circ \epsilon_{V,U} = \mathrm{id}_{U\cap V}$ for $(U, V) \in \mathcal{U} \times \mathcal{U}$.
   (c) $\epsilon_{U,U} = \mathrm{id}_U$ for $U \in \mathcal{U}$.

(Check that in fact the second condition is superfluous.) Define the notion of a morphism of $\mathcal{U}$-sheaves with descent data. Define a functor from the category of sheaves on X to the category of $\mathcal{U}$-sheaves with descent data. Prove that this functor is fully faithful. Then prove that it is an equivalence of categories. (Hint: To do this, let $\mathcal{B}$ be the base for the topology of X consisting of those open sets W which are contained in some element of the covering $\mathcal{U}$. Use the descent data to define a functor from the category of $\mathcal{U}$-sheaves with descent data to the category of $\mathcal{B}$-sheaves, and then use the previous problem.)

3. The previous problem can be reformulated as follows. Let $X_{\mathcal{U}}$ denote the disjoint union of all the elements of $\mathcal{U}$. Thus for each $U \in \mathcal{U}$, we have continuous map $q_U : U \to X_{\mathcal{U}}$, $q_U(x) = q_V(y)$ iff $U = V$ and $x = y$, and every element of $X_{\mathcal{U}}$ is $q_U(x)$ for some $U \in \mathcal{U}$ and $x \in U$. We also have a continuous map $p: X_{\mathcal{U}} \to X$. Then a $\mathcal{U}$-sheaf amounts to a sheaf on $X_{\mathcal{U}}$. Consider now the fiber product:

$$X_{\mathcal{U}} \times_X X_{\mathcal{U}} := \{(\tilde{x}_1, \tilde{x}_2) \in X_{\mathcal{U}} \times X_{\mathcal{U}} : p(\tilde{x}_1) = p(\tilde{x}_2)\}.$$

Observe that

$$X_{\mathcal{U}} \times_X X_{\mathcal{U}} = \{(q_{U_1}(x), q_{U_2}(x)) : x \in X, (U_1, U_2) \in \mathcal{U} \times \mathcal{U}\},$$

which can be written as the disjoint union of $X_{\mathcal{U}}$ (the diagonal) and the set of intersections $U_1 \cap U_2$, ranging over the set of pairs $(U_1, U_2) \in \mathcal{U} \times \mathcal{U}$ with $U_1 \neq U_2$. Now let p_1 and p_2 be the natural projections $X_{\mathcal{U}} \times X_{\mathcal{U}} \to X_{\mathcal{U}}$; similarly we have three projections

$$p_{12}, p_{13}, p_{23}: X_{\mathcal{U}} \times_X X_{\mathcal{U}} \times X_{\mathcal{U}} \to X_{\mathcal{U}} \times X_{\mathcal{U}},$$

where $p_{ij}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) := (\tilde{x}_i, \tilde{x}_j)$. Show that to give descent data on a $\mathcal{U}$-sheaf F amounts to giving an isomorphism $\epsilon: p_2^{-1}(F) \to p_1^{-1}(F)$ such that

(a) $\Delta^{-1}(\epsilon) = \mathrm{id}_F$, where $\Delta\colon X_{\mathcal{U}} \to X_{\mathcal{U}} \times_X X_{\mathcal{U}}$ is the diagonal.

(b) $p_{12}^{-1}(\epsilon) \circ p_{23}^{-1}(\epsilon) = p_{13}^{-1}(\epsilon)$.

4. Let X and T be topological spaces (or locally ringed spaces). For each open subset V of T, let $h_X(V)$ denote the set of morphism from V to X. Then h_X defines a sheaf on T. If U is an open subset of X, then h_U can be regarded as a subsheaf of h_X. Now suppose that $\mathcal{U}$ is an open covering of X, and let $h_{\mathcal{U}}(V)$ denote the union of the set of all $h_U(V)$ such that $U \in U$. That is, $h_{\mathcal{U}}(V)$ is the set of all maps $V \to X$ which factor through some $U \in \mathcal{U}$. Then $h_{\mathcal{U}}$ is a subpresheaf of h_X, but it is not a sheaf in general. Show that the sheaf associated to $h_{\mathcal{U}}$ is in fact h_X.

5. Let F be the functor from rings to sets which takes a ring A to the set of pairs (a_1, a_2) such that either a_1 or a_2 is a unit. Fix a ring A, and restrict F to the category of A-algebras of the form A_a for some $a \in A$. This functor can now be regarded as a presheaf on the basis $\mathcal{B}$ consisting of special affine open subsets of $\operatorname{Spec} A$. Let $\tilde{F}$ be the associated sheaf. What is $\tilde{F}(A)$?

6. Recall that an *idempotent* of a ring R is an element e such that $e^2 = e$. Prove that the only idempotents of a local ring are 0 and 1.

7. Let $(X, \mathcal{O}_X)$ be a locally ringed space.

   (a) Show that if $a \in \Gamma(X, \mathcal{O}_X)$, then $X_a := \{x : a(x) \neq 0\}$ is open in X.

   (b) Show that if $e \in \Gamma(X, \mathcal{O}_X)$ is idempotent, then X_e is both open and closed.

   (c) Show that if $U \subseteq X$ is open and closed, there is a unique idempotent e of $\Gamma(X, \mathcal{O}_X)$ such that $U = X_e$.

8. Let X denote the affine line with the doubled origin over a field k. Show that there is an isomorphism of functors which for every k-algebra A identifies $X(A)$ (the set of k-morphisms $\operatorname{Spec} A \to X$) with the set of pairs (a, e), where a is an element of A and e is an idempotent of $A/(a)$. (B. Poonen).

9. Let $(A_i : i \in I)$ be a family of k-algebras and let A denote the product ΠA_i. Thus for any k-algebra B, $Hom(B, A) \cong \prod Hom(B, A_i)$, and hence for any affine scheme, the natural map $X(A) \to \prod X(A_i)$ is bijective. Question: Is the true more generally, for example assuming only that X is quasi-compact and quasi-separated? (B. Poonen).

   Hints: Let $\phi_i\colon A \to A_i$ be the projection and for each i let $e_i \in A$ be the element such that $\phi_j(e_i)$ is 1 if $i = j$ and is zero otherwise. Observe that e_i is an idempotent of A, let U_i be the corresponding open and closed set, and let U be the union of all the U_i's. Show that if F is any sheaf on $S := \operatorname{Spec} A$, then $F(S) = \prod F(U_i)$. Thus the question in the previous problem asks, if X is quasi-compact and quasi-separated, is the map $F(S) \to F(U)$ bijective?

(a) Prove that U is scheme theoretically dense in S; *i.e.*, that if I is a quasi-coherent sheaf of ideals of whose restriction to U is zero, then I is zero.

(b) Prove that if V is a quasi-compact open subset of X containing U, then $V = X$.

(c) Now prove that the injectivity of $X(S) \to X(U)$ assuming X is quasi-separated.

(d) Prove the map is bijective if X is projective space.

10. Let X be a locally ringed space, let $A := \Gamma(X, \mathcal{O}_X)$, and let $S := \operatorname{Spec} A$. Show that there is a unique map of locally ringed spaces $p: X \to S$ such that $\Gamma(S, p)$ is the identity map. The locally ringed space S is an affine scheme; show that any map from X to an affine scheme factors uniquely through p.

11. Suppose that X is a quasi-compact open subset of an affine scheme S. Prove that the map $j: X \to S$ is a quasi-compact and quasi-separated morphism. Deduce that for any quasi-coherent sheaf E on X, j_*E is quasi-coherent on S.

12. Suppose that j is the inclusion of an open subscheme X into a scheme S. Prove that for any sheaf F of $\mathcal{O}_X$-modules, the natural map $j^*j_*E \to E$ is an isomorphism.

13. Let X be a scheme, let $A =: \Gamma(X, \mathcal{O}_X)$, let $S = \operatorname{Spec} A$, and let $\pi: X \to A$ be the natural map. For any sheaf E of $\mathcal{O}_X$-modules, let $\tilde{\Gamma}(E)$ be the quasi-coherent sheaf on S corresponding to the A-module $\gamma(X, E)$, and observe that there is a natural map $\pi^*\tilde{\Gamma}(E) \to E$. Suppose that X is a quasi-compact scheme. Prove that the following conditions are equivalent:

    (a) X is isomorphic to an open subscheme of an affine scheme.

    (b) For every quasi-coherent sheaf E on X, the map $\pi^*\tilde{\Gamma}(E) \to E$ is an isomorphism.

    (c) For every quasi-coherent sheaf E on X, the map $\pi^*\tilde{\Gamma}(E) \to E$ is surjective.

    (d) For every quasi-coherent sheaf of ideals I of X and every point $x \in X \setminus Z(I)$, there exists a global section $a \in \Gamma(X, I)$ such that $a(x)$ is not zero.

    (Hint: Use Hartshorne II 2.17a.)

14. Prove that every quasi-compact scheme has at least one closed point.

15. Suppose that X is a quasi-compact scheme with the property that the functor Γ from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of abelian groups is exact. Prove that X is in fact affine.

16. Let V be a valuation ring. Prove that V is coherent as a module over itself. That is, prove that V is finitely generated, and the kernel of every map $V^m \to V$ is also finitely generated.

17. Let $X \to Y$ be a separated morphism, let T be a Y-scheme, and let U be an open subset of T which is scheme theoretically dense. Prove that any two Y-morphisms from T to X which agree on U are equal.